# Orbifolds

Adrian Dawid

January 27, 2021

## Contents




**Abstract**

From representing moduli spaces to a central role in string theory: orbifolds are a vital tool in many areas of pure mathematics and mathematical physics. In this talk we aim to give an algebraic construction that will nevertheless gives us a quite tangible geometric object in the end. The talk mainly follows [Moe02] and [AK08].


## 1 Motivation

Taking quotients by groups is commonplace in mathematics. In topology the practice is justified by this well known theorem:

**Theorem 1.1.** *Let $X$ be a Hausdorff space and $G$ a topological group acting properly on $X$. Then ${}^{X}\!/_{G}$ is Hausdorff.*

However we sometimes like to work with more concrete structures than Hausdorff spaces. The nicest type of space is maybe the (smooth) manifold. And there we have:

**Theorem 1.2.** *Let $M$ be a smooth manifold and $G$ a Lie group acting smoothly, freely and properly on $M$. Then ${}^{X}\!/_{G}$ is a smooth manifold.*

Keep in mind that any discrete or finite group is a Lie group. If we compare these two theorems we will find that the property *freely* has somehow slipped into the manifold version of the theorem. And it is not there for no reason.

**Example 1.1.** Let $M = \mathbb{R}^3$ and $\mathbb{Z}_2 := {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ acting on $M$ by the proper action

$$n \bullet x = (-1)^n x.$$

Now take $X = {}^{M}\!/_{\mathbb{Z}_2}$. Can we see what this space looks like? First notice that

$$ {}^{S^2}\!/_{\mathbb{Z}_2} \cong \mathbb{RP}^2. $$

So we know our space has $\mathbb{RP}^2$ as a subspace. And we can now think of $X$ as a kind of "open-ended" cone over $\mathbb{RP}^2$ with the pointy end being $[0]$. This is clearly contractible. If it were a 3-manifold, removing a point would not change the fundamental group. But if we remove the pointy end we get something homotopic to $(0, \infty) \times \mathbb{RP}^2$. Clearly this has a non-trivial fundamental group. So $X$ is not a manifold!

However it seems wasteful not to consider spaces like the one in the example. While they are not really manifolds they seem similar enough that one might be able to save a lot of the results about manifolds. The most obvious way to study them would be to just alter the definition of a manifold so that it locally resembles ${}^{\mathbb{R}^n}\!/_{G_x}$ for a suitable group $G_x$. This process is possible but tedious and I want to present a different approach. But if we keep this idea in mind we will see that is is exactly what we end up with in the end.

# 2 Groupoid Approach

We will for now leave the realm of geometry and instead pursue a more algebraic path which will lead to an even more general and (in my opinion) elegant definition of an orbifold. In order to give this definition we will use category theory and therefore we will now recall some of its basic notions:

## 2.1 Category Theory Basics

**Definition 2.1.** A *category* $C$ consists of

1. a class of objects $\mathrm{Ob}(C)$
2. for any pair $x, y \in \mathrm{Ob}(C)$ a class of morphisms $\mathrm{Mor}_C(x, y)$
3. for any triple $x, y, z \in \mathrm{Ob}(C)$ a composition rule $\mathrm{Mor}_C(y, z) \times \mathrm{Mor}_C(x, y) \to \mathrm{Mor}_C(x, z)$ denoted by $\circ$

such that:

1. For every $x \in \mathrm{Ob}(C)$ there exists an $\mathrm{id} \in \mathrm{Mor}_C(x, x)$ s.t. for any $y \in \mathrm{Ob}(C), \phi \in \mathrm{Mor}_C(x, y)$ and $\psi \in \mathrm{Mor}_C(y, x)$ it holds $\mathrm{id} \circ \phi = \phi$ and $\psi \circ \mathrm{id} = \psi$.
2. The composition is associative.

*Remark.* As is already suggested by the fact that *set* and *class* are two distinct words they do in fact denote different concepts, *class* being the more general one. Indeed the fact that not all classes are sets is the secret weapon that makes it possible to define categories such as **Set** and **Top**, which would otherwise be ill-defined. However the usage of classes also brings with it all sorts of problem and a certain detachment from the real world (of everyday mathematics). In the following we will confine ourselves to so-called *small* categories and leave it to others to worry about the merits of *large* categories.

**Definition 2.2.** A category $C$ is called *small category* if $\mathrm{Ob}(C)$ is a set and for any $x, y \in \mathrm{Ob}(C)$ it holds that $\mathrm{Mor}_C(x, y)$ is also a set.

Given a morphism $f \in \mathrm{Mor}_C(x, y)$ we call $x$ the source and $y$ the target of $f$.

**Definition 2.3.** A (covariant) *functor* from a category $C$ to a category $\mathcal{D}$ is a mapping sending every object $x \in \mathrm{Ob}(C)$ to an object $F(x) \in \mathrm{Ob}(\mathcal{D})$ and each morphism $\psi \in \mathrm{Mor}_C(x, y)$ to a morphism $F(\psi) \in \mathrm{Mor}_{\mathcal{D}}(F(x), F(y))$ such that compositions and identities are preserved.

**Definition 2.4.** A morphism $\psi \in \mathrm{Mor}_C(x, y)$ is called an *isomorphism* if there exists a morphism $\phi \in \mathrm{Mor}_C(y, x)$ s.t. $\phi \circ \psi = \mathrm{id}_x$ and $\psi \circ \phi = id_y$.

**Definition 2.5.** A small category is called *groupoid* if all its morphisms are isomorphisms.

The name suggests a non-trivial connection to groups and indeed the next theorem will show that groupoids are in some sense a generalization of groups.

**Theorem 2.1.** *Let $(G, \circ_G)$ be a group. Then we associate with $G$ the small category $\mathcal{G}$ defined by*

$$\begin{aligned} \mathrm{Ob}(\mathcal{G}) &= \{\mathrm{pt}\} \\ \mathrm{Mor}_{\mathcal{G}}(\mathrm{pt}, \mathrm{pt}) &= G \\ \circ &= \circ_G. \end{aligned}$$

*Then $\mathcal{G}$ is a groupoid.*

*Proof.* This follows directly from the definition. □

**Theorem 2.2.** *Let $\mathcal{G}$ be a groupoid and $x \in \mathrm{Ob}(\mathcal{G})$ any object. Then $(\mathrm{Mor}_{\mathcal{G}}(x, x), \circ)$ is a group.*

*Proof.* Since $\mathcal{G}$ is a small category $\mathrm{Mor}_{\mathcal{G}}(x, x)$ is a nonempty set because $\mathrm{id} \in \mathrm{Mor}_{\mathcal{G}}(x, x)$. The group multiplication $\circ$ is associative for the same reason. We have a neutral element $\mathrm{id} \in \mathrm{Mor}_{\mathcal{G}}(x, x)$ and because $\mathcal{G}$ is a groupoid any element has an inverse. □

We can see this theorem as somehow saying that a groupoid is "locally" like a group. To get a better grip on groupoids here is a more non-trivial example:

**Example 2.1.** Define

$$\begin{aligned} \mathrm{Ob}(\mathcal{G}) &= \{\{0\}, \{1, -1\}, \{2, -2\}\} \\ \mathrm{Mor}_{\mathcal{G}}(n, m) &= \begin{cases} \{\mathrm{id}, x \mapsto -x\} & \text{if } n = m = \{0\} \\ \{\mathrm{id}\} & \text{otherwise with } n = m \\ \emptyset & \text{else} \end{cases}. \end{aligned}$$

Where the composition is given by multiplication. This groupoid can be visualized by the following graph:

$$\begin{tikzcd} 0 \arrow[loop above, "1"] \arrow[loop below, "-1"] & \pm 1 \arrow[loop above, "1"] & \pm 2 \arrow[loop above, "1"] \end{tikzcd}$$

Here we can see the local group structure quite well. But we should also keep this example in mind:

**Example 2.2.** Let $S$ be any set. Define

$$\begin{aligned} \mathrm{Ob}(\mathcal{S}) &= S \\ \mathrm{Mor}_{\mathcal{S}}(x, y) &= \begin{cases} \{\mathrm{pt}\} & \text{if } x = y \\ \emptyset & \text{else} \end{cases}. \end{aligned}$$

This is called a *discrete groupoid*.

So we see that the structure of a groupoid *per se* is rather flexible. The last fact about groupoids for now is the following definition of a kind of general quotient object that can be given a groupoid structure.

**Definition 2.6.** Let $X$ be a set and $G$ be a group acting on $X$. Then we define

$$\begin{aligned} \mathrm{Ob}(X//G) &= X \\ \mathrm{Mor}_{X//G}(x, y) &= \{(g, x) \mid x = g \bullet y\} \end{aligned}$$

We call $X//G$ an *action groupoid*.

*Remark.* Using this definition we can get a groupoid similar to example 2.1 in another way:

$$\{-2, -1, 0, 1, 2\}//{}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$$

But note that we do not get the same groupoids.

**Exercise 1.** *Refine definition 2.6 so that we do get the same groupoid.*

## 2.2 Lie Groupoids

**Definition 2.7.** Let $\mathcal{G}$ be a groupoid s.t. $G_0 := \mathrm{Ob}(\mathcal{G})$ and

$$G_1 := \coprod_{x,y \in \mathrm{Ob}(\mathcal{G})} \mathrm{Mor}_{\mathcal{G}}(x, y)$$

are topological spaces. If

1. $s : G_1 \to G_0$ mapping a morphism to its source
2. $t : G_1 \to G_0$ mapping a morphism to its target
3. the composition restricted to $\{(f, g) \in G_1 \times G_1 \mid s(f) = t(g)\} \subset G_1 \times G_1$
4. $u : G_0 \to G_1$ given by $x \mapsto \mathrm{id}_x$
5. $i : G_1 \to G_1$ mapping a morphism to its inverse

are all continuous we call $\mathcal{G}$ a *topological groupoid*.

**Definition 2.8.** Let $\mathcal{G}$ be a topological groupoid, then the group $\mathrm{Mor}_{\mathcal{G}}(x, x)$ denoted by $G_x$ is called the *local isotropy group* at $x$.

*Remark.* As a subspace of $G_1$ the group $G_x$ inherits a subspace topology and is thus a topological group.

**Definition 2.9.** A Lie groupoid is a *topological groupoid* where $G_0$ and $G_1$ carry the structure of a smooth manifold and all the above maps are smooth and $s, t$ are smooth submersions.

**Definition 2.10.** Let $\mathcal{G}$ be a Lie groupoid. It is called

1. *proper groupoid* if $(s, t) : G_1 \to G_0 \times G_0$ is proper.
2. *foliation groupoid* if $G_x$ is discrete for every $x \in G_0$.
3. *étalé groupoid*[1] if $s$ and $t$ are local diffeomorphisms.

**Definition 2.11.** Let $\mathcal{G}$ be an étalé groupoid. Then

$$\dim(\mathcal{G}) = \dim(G_1) = \dim(G_0)$$

is called the *dimension* of $\mathcal{G}$.

**Corollary 2.2.1.** *Let $\mathcal{G}$ be an étalé groupoid. Then $\mathcal{G}$ is a foliation groupoid.*

*Proof.* This follows directly. We note that $G_x = s^{-1}(x) \cap t^{-1}(x)$. Since $s$ and $t$ are local isomorphisms we get that $G_x$ consists of isolated points, i.e. it carries the discrete topology. □

**Corollary 2.2.2.** *Let $\mathcal{G}$ be a proper foliation groupoid, then $G_x$ is finite for any $x \in G_0$.*

*Proof.* This proof is just a simple. Since $\mathcal{G}$ is a foliation groupoid we know that $G_x$ is discrete. Because $\{x\}$ is compact since $G_0$ is a (smooth) manifold we know that $s^{-1}(x) \cap t^{-1}(x) = G_x$ is compact. A discrete and compact topological space must be finite, thus $G_x$ is finite. □

[1] In French the verb *étaler* means *to spread out something* as in *étaler du beurre sur du pain*. The term étalé used as an adjective is the past participle of *étaler*. Sometimes an étalé groupoid is called an "étale groupoid" in English sources. In my opinion this should be avoided as an étalé groupoid is a special case of étalé space (*espace étalé* in French). Étale is actually a different French word that has a related but different mathematical meaning and is used for morphisms but not for spaces.

## 2.3 Orbifold Structures and Orbifolds

**Definition 2.12.** An *orbifold groupoid* is a proper foliation groupoid.

**Corollary 2.2.3.** *A proper étalé groupoid is an orbifold groupoid.*

However a groupoid is not a topological space and we thus need a further definition.

**Definition 2.13.** Let $\mathcal{G}$ be an orbifold groupoid then

$$|\mathcal{G}| = {}^{G_0}\!\big/_{x \sim y \forall y \in t(s^{-1}(x))}$$

is called the *orbit space*.

**Definition 2.14.** Let $X$ be a paracompact Hausdorff space and $\mathcal{G}$ an orbifold groupoid and $f : |\mathcal{G}| \to X$ be a homeomorphism. The triple is called an *orbifold structure*.

And finally we can give the following definition:

**Definition 2.15.** An *orbifold* $\mathcal{X}$ is a paracompact Hausdorff space with an equivalence class[2] of orbifold structures. A specific structure is called a *presentation* of $\mathcal{X}$.

We directly get two theorems, which show us that our definition does actually fulfill what we want based on our intuition of what an orbifold should be.

**Theorem 2.3.** *Let $M$ be a (smooth) n-manifold. Then we can naturally associate an orbifold structure $\mathcal{M}$ with it that makes $M$ into an orbifold.*

*Proof.* Let $\mathcal{M}$ be the discrete groupoid associated with the underlying set of $M$. Then $M_0 = \mathrm{Ob}(\mathcal{M})$ naturally is a smooth manifold. By construction there is a bijection

$$M_1 := \coprod_{x,y \in \mathrm{Ob}(\mathcal{M})} \mathrm{Mor}_{\mathcal{M}}(x, y) \cong \coprod_{x \in \mathrm{Ob}(\mathcal{M})} \mathrm{Mor}_{\mathcal{M}}(x, x) \cong M_0$$

thus making $M_1$ into a smooth manifold as well. Then $s, t, u, i$ are the identity map and thus smooth. We still have to look at $\circ$ restricted to

$$\{(f, g) \in M_1 \times M_1 \mid s(f) = t(g)\} = \Delta \subset M_1 \times M_1.$$

However this is just the projection to either component and thus also smooth. So we have that $\mathcal{M}$ is an étalé groupoid since $s$ and $t$ are diffeomorphisms. Since $(s, t) : M_1 \to M_0 \times M_0$ is just the diagonal map and thus proper it is actually an orbifold groupoid. We define

$$f : |\mathcal{M}| \to M$$

using the canonical identification which is a homeomorphism. Thus $[\mathcal{M}]$ makes $M$ into an orbifold. □

Now we will encounter a way to get more interesting orbifolds. But first we need a new definition:

**Definition 2.16.** Let $G$ be a group acting on a set $X$. The action is called *almost free* if

$$\{g \in G \mid g \bullet x = x\}$$

is finite for any $x \in X$.

**Corollary 2.3.1.** *Let $G$ be a finite group acting on a set $X$. Then $G$ acts almost freely.*

**Corollary 2.3.2.** *Let $G$ be a discrete topological group acting properly on a topological space[3] $X$. Then $G$ acts almost freely.*

[2]we don't have a notion of equivalence for orbifold structures yet, but let's not quibble, we will get there.

[3]A space in which $\{x\}$ is compact for any $x \in X$ is enough. And that is always true even if $X$ is not Hausdorff and horrible in a myriad of other ways. So a topological space is really the correct notion here.

So we see that while the usage of the term almost is in line with its usual usage (i.e. there are only finitely many exceptions) the condition of an action being almost free is usually trivial in our cases whereas being free is a quite strong condition.

**Theorem 2.4.** *Let $M$ be a (smooth) n-manifold and $G$ a Lie group acting almost freely and properly on $M$. Then $G//M$ defines a natural orbifold structure on ${}^{M}\!/_{G}$.*

The word Lie group is nice but in many cases not really necessary, as can be seen by these corollaries:

**Corollary 2.4.1.** *Let $M$ be a (smooth) n-manifold and $G$ a discrete group acting properly on $M$. Then $G//M$ defines a natural orbifold structure on ${}^{M}\!/_{G}$.*

We will only show this special case.

*Proof.* Denote $X := {}^{M}\!/_{G}$ and $\mathcal{X} = G//M$. Then $X_0 \cong M$ and

$$X_1 := \coprod_{x,y \in \mathrm{Ob}(\mathcal{X})} \mathrm{Mor}_{\mathcal{X}}(x,y) = G \times M.$$

with $s$ begin the projection to $M$ and $t$ begin the action. This is a Lie groupoid. We now have to check that it is a proper foliation groupoid. The proper part directly follows from the action of $G$ begin proper. Now since $G$ acts almost freely we have that $G_x$ is finite for any $x \in M$ thus we have a foliation groupoid and thus an orbifold groupoid. Now we further see

$$|\mathcal{X}| \cong {}^{X_0}\!/_{x \sim y \forall y \in t(s^{-1}(x))} \cong {}^{X_0}\!/_{G} \cong {}^{M}\!/_{G}.$$

Thus the canonical identification gives us a homeomorphism $f : |\mathcal{X}| \to {}^{M}\!/_{G}$. This defines an orbifold structure and thus makes ${}^{M}\!/_{G}$ into an orbifold. This concludes our proof. □

**Corollary 2.4.2.** *Let $M$ be a (smooth) n-manifold and $G$ a finite group acting properly on $M$. Then $G//M$ defines a natural orbifold structure on ${}^{M}\!/_{G}$.*

## 2.4 Equivalence of Orbifold (Structures)

Until now we don't really know what the term "equivalence class of orbifold structures" really is supposed to mean. It does not mean the equivalence of categories. Such a notion would not respect our smooth structures (the infamous axiom of choice can be used easily to come up with counterexamples). The notion we actually need is called a *Morita equivalence*.

**Definition 2.17.** Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A functor $F : \mathcal{G} \to \mathcal{H}$ is called a *homomorphism* if $F_0 : G_0 \to H_0$ and $F_1 : G_1 \to H_1$ are smooth.

**Definition 2.18.** A homomorphism $F : \mathcal{G} \to \mathcal{H}$ of Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ is called an *equivalence* if

1. $t \circ \pi_2 : \{(g,x) \in G_1 \times H_0 \mid s(g) = F(x)\} \to G_0$ is a surjective submersion.
2. the diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{F} & G_1 \\ {\scriptstyle (s,t)}\downarrow & & {\scriptstyle (s,t)}\downarrow \\ H_0 \times H_0 & \xrightarrow{F \times F} & G_0 \times G_0 \end{array}$$

   is commutative and $H_1$ together with $F$ and $(s,t)$ is a fibered product[4] of smooth manifolds.

**Exercise 2.** *Go through these notes and find the many places where using a fibered product would allow a more elegant reformulation.*

[4]This is a general notion in categroy theory and not (directly) connected to fiber bundles [Mat].

We should note that this definition does not look symmetric and that is because it is not. So the name "equivalence" should be taken with a grain of salt. It is probably best to ignore this definition whenever possible, but nevertheless it should be given for the sake of completeness. We now introduce a more symmetric term:

**Definition 2.19.** Let $\mathcal{G}, \mathcal{G}'$ and $\mathcal{H}$ be Lie groupoids with equivalences

$$\mathcal{G} \longleftarrow \mathcal{H} \longrightarrow \mathcal{G}'.$$

Then $\mathcal{G}$ and $\mathcal{G}'$ are called *Morita equivalent*.

We give the following important result without proof:

**Theorem 2.5.** *A Lie groupoid is a proper foliation groupoid if and only if it is Morita equivalent to a proper étalé groupoid.*

Now we can actually define the term "equivalent orbifold structures".

**Definition 2.20.** Two orbifold structures $f : |\mathcal{G}| \to X$ and $g : |\mathcal{H}| \to X$ are called equivalent if there is an equivalence $\mathcal{G} \to \mathcal{H}$.

**Corollary 2.5.1.** *Any orbifold has a presentation as a proper étalé groupoid.*

Among other things this allows us to define the dimension of an orbifold.

# 3 Examples

Now we can look at some examples.

**Example 3.1.** By corollary 2.4.2 our initial example ${}^{\mathbb{R}^3}\!/_{\mathbb{Z}_2}$ has an orbifold structure induced by $\mathbb{R}^3//\mathbb{Z}_2$.

The next example is the so called *Kummmer surface*. It is not a surface in the topological sense but it is an orbifold of dimension 2.

**Example 3.2.** We define the $\mathbb{Z}_2$ action on $\mathbb{T}^4$ by

$$1 \bullet (z_1, z_2, z_3, z_4) = (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}).$$

Then we have an orbifold

$$\mathcal{K} = {}^{\mathbb{T}^4}\!/_{\mathbb{Z}_2} \cong {}^{S^1 \times S^1 \times S^1 \times S^1}\!/_{\mathbb{Z}_2}$$

with the structure induced by $\mathbb{T}^4//\mathbb{Z}_2$.

**Example 3.3.** Consider $S^{2n+1} \subset \mathbb{C}^{n+1}$ and the group action of $S^1 \subset \mathbb{C}$ by

$$\varphi \bullet (z_0, \dots, z_n) = (\varphi^{a_0} z_0, \dots, \varphi^{a_n} z_n)$$

where $a_0, ..., a_n \in \mathbb{Z}$. Then

$$W\mathbb{P}(a_0, \dots, a_n) := {}^{S^{2n+1}}\!/_{S^1}$$

is an orbifold called the *weighted projective space*.

**Example 3.4.** The orbifold $W\mathbb{P}(1, a)$ is called a teardrop orbifold.

# 4 Outlook

With more time than 90 minutes a lot of structures could be established on orbifolds. The presented algebraic approach makes this straight forward in many cases. The list of such structures includes (but is not limited to):

- The Euler characteristic
- Homology and Cohomology
- Fundamental groups
- Bundles over orbifolds

The theory can also be extended to a functional analytic setting giving rise to so called *polyfolds* [HWZ17].

# 5 Further Reading

If you are interested in the historical definition of an orbifold the original paper by Satake is very useful. The more geometric approach is given beautifully in chapter 13 of Thurston's book:

[Sat56] I. Satake. "On a Generalization of the Notion of Manifold". eng. In: *Proceedings of the National Academy of Sciences - PNAS* 42.6 (1956), pp. 359–363. ISSN: 0027-8424.

[Thu02] William Thurston. *Geometry and topology of three-manifolds*. 2002. URL: http://library.msri.org/books/gt3m/.

More information on the (Lie) groupoid approach can be found in the (short) introduction by Moerdijk. A lot more information (in particular concerning applications in string theory) can be found in the book by Aden et al.:

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and Stringy Topology*. eng. Vol. 171. Cambridge tracts in mathematics. Cambridge: Cambridge University Press, 2007. ISBN: 0521870046.

[Moe02] Ieke Moerdijk. "Orbifolds as groupoids: an introduction". In: *Orbifolds in mathematics and physics (Madison, WI, 2001)*. Vol. 310. Contemp. Math. Amer. Math. Soc., Providence, RI, 2002, pp. 205–222. DOI: 10.1090/conm/310/05405.

More generally a book I like on category theory is the following:

[Rie16] Emily Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2016. URL: https://math.jhu.edu/~eriehl/context.pdf.

# 6 References


[AK08] Alejandro Adem and Michele Klaus. *Lectures on Orbifolds and Group Cohomology*. 2008. URL: http://www.math.ubc.ca/~adem/hangzhou.pdf.

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and Stringy Topology*. eng. Vol. 171. Cambridge tracts in mathematics. Cambridge: Cambridge University Press, 2007. ISBN: 0521870046.

[HWZ17] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder. *Polyfold and Fredholm Theory*. 2017, pp. 225–257. URL: https://arxiv.org/pdf/1707.08941.pdf.

[Jr19] Francisco C. Caramello Jr. *Introduction to orbifolds*. 2019. URL: https://arxiv.org/pdf/1909.08699.pdf.

[Mat] Encyclopedia of Mathematics. *Fibre product of objects in a category.* URL: http://encyclopediaofmath.org/index.php?title=Fibre_product_of_objects_in_a_category&oldid=30737.

[Moe02] Ieke Moerdijk. “Orbifolds as groupoids: an introduction”. In: *Orbifolds in mathematics and physics (Madison, WI, 2001)*. Vol. 310. Contemp. Math. Amer. Math. Soc., Providence, RI, 2002, pp. 205–222. DOI: 10.1090/conm/310/05405.

[Rie16] Emily Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2016. URL: https://math.jhu.edu/~eriehl/context.pdf.

[Thu02] William Thurston. *Geometry and topology of three-manifolds*. 2002. URL: http://library.msri.org/books/gt3m/.